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Abandoning Standard Assumptions in Pareto Optimization Models: Simulating Market Scenarios with Incorrectly Estimated Preference Structures

Joshua Bernard

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Joshua Bernard

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Dr. Kevin Rion, Thesis Director
Dr. Irina Seceleanu, Committee Member
Dr. Laura Gross, Committee Member

Abstract

This honors thesis examines the consequences of abandoning specific underlying assumptions of economic models used to describe the distribution of goods among individual agents or parties and the information about each one's preferences. In microeconomic theory, the Edgeworth Box, Pareto-optimal trade, and convex (especially Cobb-Douglas) preference structures are used to model the process in which consumers and producers make trade-off choices that allocate limited resources among competing agents. This thesis investigates the common underlying assumptions of these economic models by drawing upon mathematical theory to develop both an analytical framework and the tools that help us establish boundaries for these economic problems. The means of investigation involves extensive use of mathematical reasoning and computer simulation. The main focus of this investigation is to determine the consequences of relaxing the theoretical assumption stating that agents participating in Pareto-efficient exchange always operate with complete and correct information. The objective is to first determine the changes in Pareto optimization and price-setting that occur as a result of differences in perception regarding marginal rates of exchange and then to determine which trades are and are not Pareto-efficient.

Establishing Preference Structure through Utility Functions

The field of microeconomics studies individual economic behavior, which entails individuals making trade-offs in order to distribute limited or scarce recourses among other individual agents. A variety of models have been produced to describe how individual consumers and producers make choices based on the established preferences of each agent, and given the constraints imposed by the limited availability of resources. It is generally assumed that in light of their preferences, consumers and producers choose from the set of affordable alternatives the one that will maximize their total satisfaction received from consuming or producing goods and services. The array of preferences of the agents can be modeled mathematically by a non-negative real valued utility function U: $[0, \infty)^n \rightarrow [0, \infty)$, so that any combination of quantities of *n* goods consumed or produced (referred to going forward as a *bundle*) will be mapped to an assigned numerical value representing the resulting level of satisfaction, or utility, incurred. Cobb-Douglas functions, of the form $\mathrm{U}(\boldsymbol{X}) = \prod_{i=1}^{n}(x_i)^{\alpha_i}$, whose parameters α_i are non-negative numbers which sum to one, are standard examples of such utility functions. This type of Cobb-Douglas function has an ideal functional form to work with because it has the following properties:

- U is a strictly monotonically increasing function. For every x_i, the function mapping x_i to $\mathrm{U}(x_1, \dots, x_i, \dots, x_n)$ is strictly increasing.
- U is continuous and differentiable; this generally only applies to the nonnegative domain (except when one of the values of x_i is zero), but in this context that condition is usually not relevant.
- The preference relation induced by U is transitive. Specifically, the relation '*is preferred to'* that is defined by "bundle **X** is preferred to bundle **Y** if and only if U(**X**) ≥ U(**Y**)" is transitive. This is a trivial result of the monotonicity of the function and the standard ordering of the real domain and codomain.
- The model observes the law of diminishing returns, also known as the law of diminishing marginal utility. The marginal utility of a good x_i is the additional utility gained for every additional unit of x_i acquired. Mathematically, the marginal utility MU_{x_i} of x_i is the partial derivative of U with respect to x_i. Let Δx_i denote an increase in good *i* consumed or produced, so that some bundle *A* is represented by the vector

$(x_1, \dots, x_i, \dots, x_n)$ and *B* represented by $[x_1, \dots, (x_i + \Delta x_i), \dots, x_n]$. Then, by the law of diminishing returns, for every variable x_i, if Δx_i denotes an increase in commodity *i* consumed or produced, then holding the quantities of other goods *j* constant, the marginal gain from increasing x_i is less when bundle *B* is owned than when *A* is owned (i.e., $\mathrm{MU}_{\mathrm{x_i}}(x_1, \dots, x_i, \dots, x_n) \geq \mathrm{MU}_{\mathrm{x_i}}(x_1, \dots, (x_i + \Delta x_i), .., x_n)$). Featured in Figure 1 is a graph of a cross-section of a bi-variable Cobb-Douglas function with a fixed *y* value; for all *x* > 0, as *x* increases, U increases, but at a decreasing rate, as evidenced by the graphed curve's downward concavity.

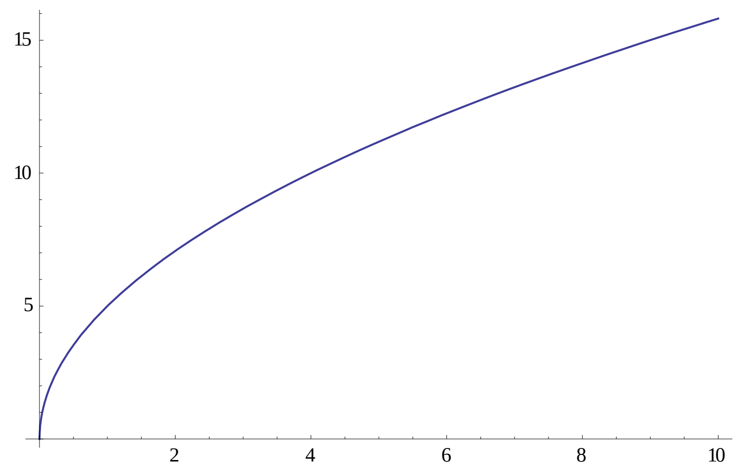


Figure 1. A cross section of $\mathrm{U}(x, y) = \sqrt{x * y}$ with $y = 25$.

- Returns to scale are constant; this is often described as a situation where doubling of all quantities consumed or inputs will result in doubling utility or output, respectively (Nicholson and Snyder 226). This is a result of the parameters α_i summing to one (Fuleky).
- For constant returns to scale, the marginal utilities are never negative and the utility function does not attain a maximum**.** For each commodity *i*, the marginal utility of x_i decreases and converges to zero as its *i*th argument increases without bound.

- Horizontal cross sections (or contours) of the surfaces plotted by this family of functions are convex with respect to the origin of the domain space. Any line segment whose endpoints are two points lying on such a cross section will always remain inside the region whose lower bound is the cross section (Boyd and Vandenberghe 67). An example of what such a cross section looks like can be seen in Figure 2 below.

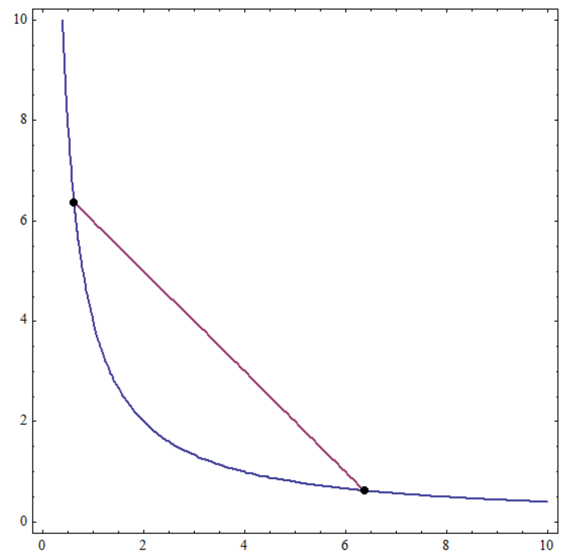


Figure 2. This contour of a Cobb-Douglas function is a convex curve.

Isoquants are sets of bundles **X** on the domain of the utility function which yield equal amounts of utility U_α: $I_\alpha = \{\boldsymbol{X} \in [0, \infty)^n : \mathrm{U}(\boldsymbol{X}) = U_\alpha\}$. The individual, given a choice between two bundles on some set I_α, is indifferent between each, since neither is considered preferable to the other. Thus, isoquants are often called “indifference curves” (Nicholson and Snyder 60). Figure 3 provides an example of a contour graph of a bi-variable Cobb-Douglas function displaying these curves; the curves shown represent sets of the graphed surface for which the vertical coordinate, corresponding to utility, stays constant (hence the name “isoquant”). Since the function is monotonically increasing in all directions, contours further from the origin correspond to higher levels above the XY plane.

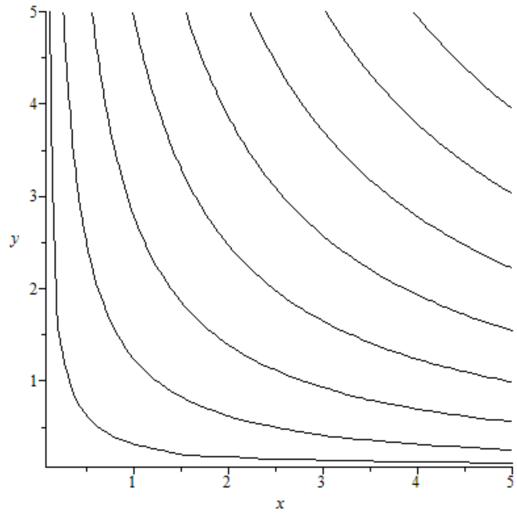


Figure 3. A contour graph of $\mathrm{U}(x, y) = \sqrt{x * y}$.

While each bundle on an isoquant will each yield identical degrees of utility, different bundles are associated with different marginal rates of substitution; a marginal rate of substitution between two goods is defined as the quantity of one commodity *i* that an individual is willing to trade for a unit of some other commodity *j*. In the special case where there are only two commodities available for consumption, the marginal rate of substitution between two goods *x* and *y* is generated by taking the ratio between two first-order partial derivatives of the relevant utility function (where the notation for each partial derivative is MU, for "marginal utility"); $\frac{-\mathbf{MU_x}}{\mathbf{MU_y}} = -\frac{\frac{\partial \mathrm{U}}{\partial \mathrm{x}}}{\frac{\partial \mathrm{U}}{\partial \mathrm{y}}} = -\frac{\mathrm{dy}}{\mathrm{dx}}$, by the chain rule (Nicholson and Snyder 62); the motivation for this is rooted in the notion that an individual will want to optimize the proportion of the desirability of additional y relative to that of additional units of x. The resulting $\frac{\mathrm{dy}}{\mathrm{dx}}$ is then representative of the ratio between instantaneous marginal utilities of goods x and y, and is also the effective slope, or rate of change, between *x* and *y*, holding U constant. In the models used here, the convexity of the isoquants ensures that the marginal rate of substitution tends to be more extreme for more unbalanced bundles.

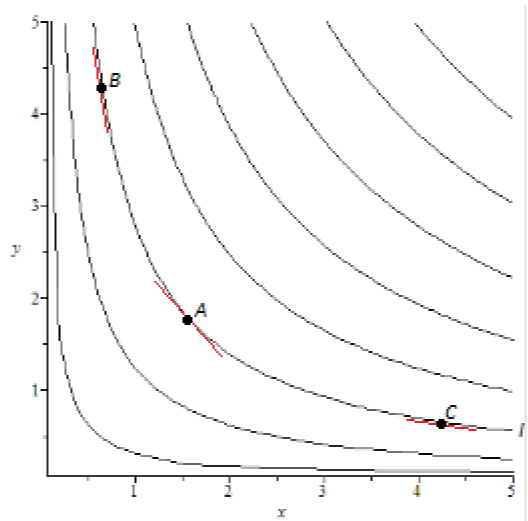


Figure 4. The convexity of the isoquants ensures that extreme bundles result in extreme marginal rates of exchange.

This can be seen in Figure 4 above, where the relative steepness of the line tangent to the highlighted isoquant *I* at bundle *B*, which has much more of *y* than *x* (so that *x* becomes more desirable), and relative flatness of the one tangent to *I* at *C*, which has much more of *x* than *y* (making *y* more desirable), stand in contrast to the tangent line at balanced bundle *A*, whose slope's absolute value is close to 1, corresponding to a one-to-one relationship between the relative exchangeability between *x* and *y*.

When examining individual consumer or producer utility, if price levels of all goods are already determined (if not at least relative price levels, in situations where there is no purely monetary medium of exchange), then a linear constraint can be established whose upper bound is determined by how much "currency" the individual has, and whose steepness is determined by the relative market prices between each pair of exchangeable goods.

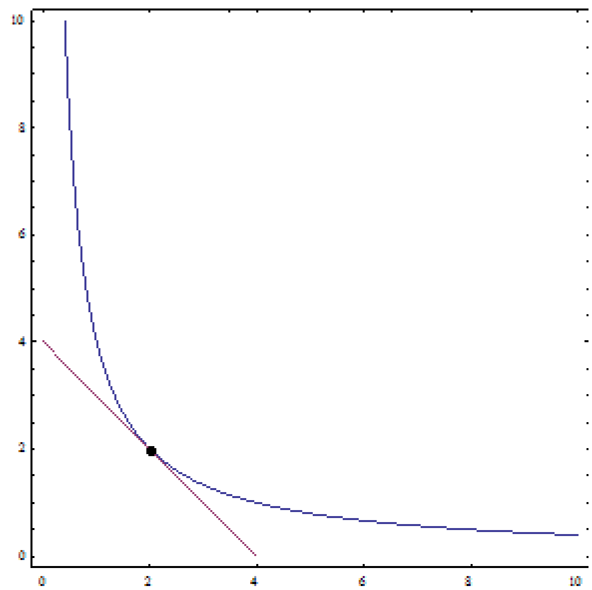


Figure 5. A linear budget constraint, and the optimal isoquant with which it intersects.

In the simple case with a market of only two goods, as seen in Figure 5, this constraint, which acts as a budget constraint for a consumer and a type of production frontier for a producer, has a boundary whose slope takes the form of the price ratio between goods x and y: $\frac{-\mathbf{P_x}}{\mathbf{P_y}}$ (Nicholson and Snyder 69).

Given an economy in which at least one good functions as some form of commodity money, or given that there is some form of currency, each commodity *i*'s unit price $\mathbf{P_{x_i}}$ can be defined independently of other goods *j* (with the exception of the commodity *k* playing the role of commodity money if truly liquid cash does not exist). If the above axiom holds, then consumer equilibrium is met when the ratio of marginal utility of x_i to each price of x_i is the same for every x_i:

$$\frac{\mathbf{MU_{x_1}}}{\mathbf{P_{x_1}}} = \frac{\mathbf{MU_{x_2}}}{\mathbf{P_{x_2}}} = \cdots = \frac{\mathbf{MU_{x_{n-1}}}}{\mathbf{P_{x_{n-1}}}} = \frac{\mathbf{MU_{x_n}}}{\mathbf{P_{x_n}}}$$

This then implies that for each pair of commodities *i* and *j*, $\frac{\mathbf{MU_{x_i}}}{\mathbf{P_{x_i}}} = \frac{\mathbf{MU_{x_j}}}{\mathbf{P_{x_j}}}$**,** which immediately results in $\frac{\mathbf{P_{x_j}}}{\mathbf{P_{x_i}}} = \frac{\mathbf{MU_{x_j}}}{\mathbf{MU_{x_i}}}$, for each *i* and *j* available in the market (Nicholson and Snyder 73). Thus, as shown in Figure 4, consumer equilibrium is satisfied if and only if that person has acquired a bundle of goods (e.g. the solid black dot in Figure 4 for which (*x*,*y*)=(2,2)) for which the boundary of the individual's constraint is perfectly tangential to the corresponding indifference curve.

The Edgeworth Box as a Model for Trade

The earlier assumption of established price levels ends up begging the question of how they might be established in the first place. This is where the Edgeworth box model becomes relevant; rather than pitting the individual's preference contours against some perfectly linear budget constraint, one could instead pit it against those of another individual (or those of many, in the general case). Then the marginal rates of exchange for the second individual are like starting price levels for trade from which the first can negotiate. The Edgeworth box is used to model the allocation of goods resulting from negotiation between or among parties. There exist a fixed number of active participants *m* in some closed economy, where there are a fixed number of goods available for distribution *n* (In the simplest case, the one which receives the most consideration here due to its relative simplicity, (*m*,*n*) = (2,2).). All *m* parties start with initial bundles $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_m}$, where $\boldsymbol{A_i} = \left(a_{ix_1}, a_{ix_2}, \ldots, a_{ix_n}\right)$, for each *i*th individual (for the purposes of this project, assume that for each individual, the initial bundle is generated from the tangency between some production utility function and some production possibilities frontier behaving as a productive linear budget constraint). Furthermore, the total amount of each good available, at least initially, is assumed to be determined by the (vector) sum of initial bundles. For any given trade, all reallocations of goods between parties are Pareto-efficient; redistribution occurs among some subset of the group of parties in the economy only if it is believed or known that marginal utility of redistribution is nonnegative for both/all trading parties in that group of agents. Given that each party's preference structure satisfies the Cobb-Douglas properties outlined earlier, this means that reallocation occurs only if no party has to give up anything, or if each party gets back more than (s)he would have to give up. Every individual is assumed to be able to negotiate and trade with every other individual with no transaction costs or barriers to entry.

The space in which this trading activity occurs can be described as a cross-product of the domain spaces of the utility functions U_i, where vector $\mathbf{A} = (\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_m})$ lies on this cross product and represents all combinations of quantities possessed by every individual. Figure 5 shows an Edgeworth box modeling the potential trading behavior in a closed two-person economy with only two commodities available for trade. Given some initial allocation $(\boldsymbol{A_1}, \boldsymbol{A_2})$ and utility functions U_1 and U_2, the Edgeworth box is set up so that the isoquants I_1 (corresponding to the utility level of $U_1(\boldsymbol{A_1})$) and I_2 (corresponding to the utility level of $U_2(\boldsymbol{A_2})$) are heuristically diametrically arranged, forming a convex blue region, whose two boundaries intersect at $(\boldsymbol{A_1}, \boldsymbol{A_2})$. This illustrates that anything owned by one individual is out of the reach of the other without some sort of trade. Also, $\boldsymbol{A_1}$, if presented near the upper-right corner of the box, implies the first individual is wealthy while the second individual is considered wealthier when $\boldsymbol{A_2}$ is near the bottom-left corner. The interior of the convex blue region is the region of reallocation points for which utility increases for both parties, and its boundaries consist of reallocations for which one party's net gains are nearly maximized and for which the other party's utility stays constant.

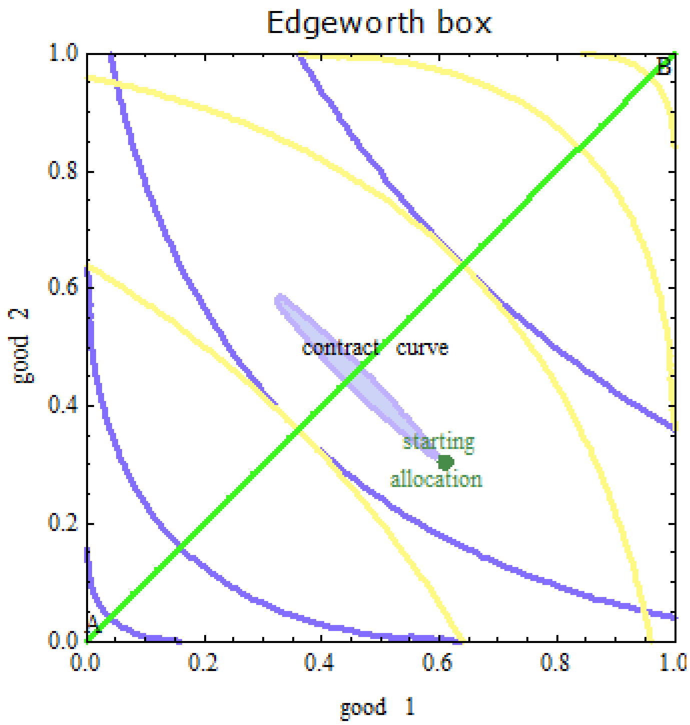


Figure 6. An Edgeworth box with two participants, and two goods available for trade.

As a result of the convexity of the individuals' preference structures, any allocation of goods can thus only converge inside the interior of the subspace of the Edgeworth box bounded by isoquants $I_{\propto_i}$ corresponding to the current allocations $\boldsymbol{A_i}$ (Autor). Here, it is assumed that initially, if the quantity 1 represents 100% of any good *j* available, then it must hold that for every such *j*, for each individual *i*, $\sum_{i=1}^{m} a_{ix_j} = 1$ (that is, the sum of all quantities owned by every individual is 100% of that good available in the market); perhaps more importantly this condition holds when reallocation is Pareto-optimal. This allows $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ to overlap on the two-dimensional Edgeworth box at all times, ensuring that if total outputs Q_x (quantity produced of good *x*) and Q_y (quantity produced of good *y*) are known, then $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are complements such that knowing one is sufficient to compute the other (i.e., $\boldsymbol{A_2} = (Q_x, Q_y) - \boldsymbol{A_1}$).

Since the availability of all unclaimed goods are exhausted, in order for one person to gain it would require at least one other to give something up. When Pareto-efficient trade or reallocation is no longer possible, the distribution of goods is considered Pareto-optimal (Nicholson and Snyder 362). The set of all Pareto-optimal allocations in an Edgeworth box is known as a contract curve (such as the bright green curve in Figure 6), defined as the set of all allocations $(\boldsymbol{A_1}, \boldsymbol{A_2}, \dots, \boldsymbol{A_m})$ for which one of the *i*th individual's marginal utility MU_{i_a} can be positive only if there is some other *j*th individual for which at least one of that individual's marginal utilities MU_{j_b} is negative. The contract curve can also be defined as the set of all intersections between mutually optimal, and thus tangential, isoquants (the only nonempty intersections by definition), written $\boldsymbol{C} = \{(\bigcap_{i=1}^{m} I_{\propto_i}) : (\bigcap_{i=1}^{m} I_{\propto_i}) \neq \emptyset\}$ (Nicholson and Snyder 362-363). A few of these tangencies can be seen in Figure 6.

The aforementioned "set of nonempty intersections of isoquants" definition of the contract curve is important, because the contract curve must then be a set which determines the range of possible equilibrium price levels between goods. Similar to the tangency between the optimal isoquant against a budget constraint featured in Figure 5, the slope of the line tangent to isoquants in an Edgeworth box after Pareto optimization should determine relative price levels between goods. Now, let this miniature economy's market of interest be a free one, such that no transaction produces externalities, the market is perfectly competitive, there are no transaction costs, and all participants are fully informed. Then according to the First Welfare Theorem, if those conditions hold, when the market is in equilibrium, it must also be Pareto efficient (Autor). Furthermore, the Second Welfare Theorem states that if preferences are convex (as outlined earlier), Pareto efficient allocations can be market equilibria (Autor). The result of these assumptions is that the set of available Pareto improvements (the blue space in the above example), given some initial allocation, will contain some subset of market equilibria. Since the contract curve is defined as the set of all allocations for which relative prices between goods are agreed upon, the set of Pareto optimal allocations must then equal the set of allocations for which market equilibria (and thus universally agreed-upon price levels) are established.

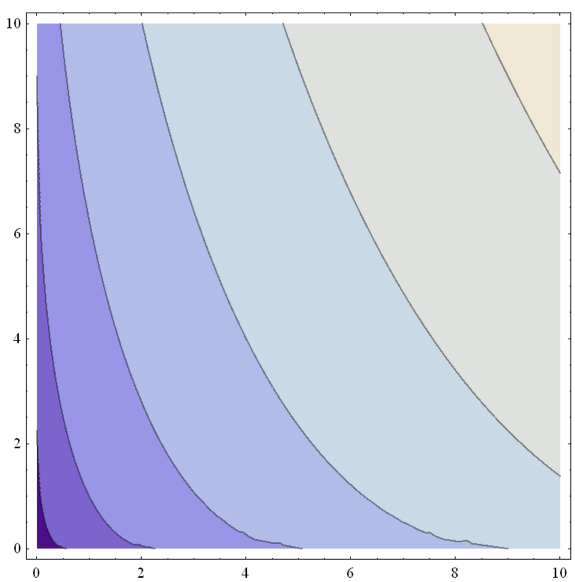
Consider a scenario in which two individuals gather or produce two types of resources in a Robinson-Crusoe-style closed economy where by definition each individual acts as both consumer and producer (Miron). If the conditions required by the First Welfare Theorem are satisfied, this will result in the two individuals negotiating with each other, and ultimately trading such that a Pareto-optimal reallocation is the result, and equilibrium (relative) market price levels are established. This, of course, would then allow an outside observer to monitor changes in quantities produced and in price levels for each good after each cycle of consumption and production. This grants sufficient information needed to calculate this economy's periodic nominal GDP, which is the sum of products of quantity and price level of each good, written $\text{GDP} = \sum P_i \cdot Q_i$ (Mankiw 21). Tracking GDP, furthermore, allows for monitoring of economic growth and recession, given that one could adjust it for inflation. By the Keynesian IS-LM model, GDP, when in equilibrium with planned aggregate expenditures, can be considered equivalent to aggregate income so that whenever the GDP is decreasing recession is more likely, and that increasing GDP implies probable recovery or growth (Mankiw 275).

However, this is all assuming that no individual is interested in any economic activity other than completely honest and procedurally fair behavior. Neither individual is assumed to be willing to lie, cheat, or steal so that (s)he is at an unfair advantage. At least initially, it is not assumed that each individual is interested in gaining, then exploiting, an inequitable amount of wealth or market power. Depending on each individual's preference structure and productivity, however, an initially free market could easily become unbalanced over time; after all, if for some reason the negotiation procedure preceding reallocation repeatedly results in one individual gaining more than the other, while each negotiation session yields a Pareto-optimal result, it will not result in any substantial redistribution of wealth, if any at all. As a matter of fact, a scenario in which one individual has everything and the other has nothing, Pareto optimality is satisfied (ignoring the issues raised by possible violations of continuity and differentiability at the extreme corners of the Edgeworth box, assume the contract curve includes these corners) (Autor). Thus, the wealthy individual will not agree to trade anything with the poor one unless the poor one can offer something perceived to be of at least equal value. Also, an initially free market, given individual production frontiers for which the cost of producing x for one individual is relatively low and the cost of producing y is relatively low for the other individual, can be expected to result in an asymmetric market where each individual ends up specializing in the production of one commodity because the differences in costs of production create comparative advantages in production.

Simulating Pareto Optimization with Mathematica

In order to answer questions about the short-term and long-term behavior of individuals with Cobb-Douglas preferences always willing to trade in a Pareto-efficient manner, attempts were made to develop an algorithm to simulate the production and consumption behaviors of such individuals. In this simulation programmed using Mathematica, a set of initial conditions for each individual is established, and for the sake of simplicity, the simulation is limited to a closed two-person economy in which only two commodities can be produced and are available for consumption. Each individual is assigned two utility functions, one of which is associated with utility of production, and the other associated with consumer utility, each respectively satisfying the form $\mathrm{C}(\boldsymbol{X}) = A \cdot \prod_{i=1}^{n}(x_i)^{\alpha_i}$ and $\mathrm{P}(\boldsymbol{X}) = B \cdot \prod_{i=1}^{n}(x_i)^{\beta_i}$, with parameters A, B, each α_i, and each β_i being randomly generated.

Unfortunately, due to problems arising from the need to debug the program before it works properly, and because of time constraints, at the time of writing the desired recursive program remains unfinished. An already existing Mathematica program modeling the Edgeworth box, written by Seth J. Chandler, was found, and for reasons yet unknown uses a slightly different functional form, $U(x, y) = a\sqrt{x} + b\sqrt{y}$ (the image produced for Figure 6 was generated by Chandler's program).



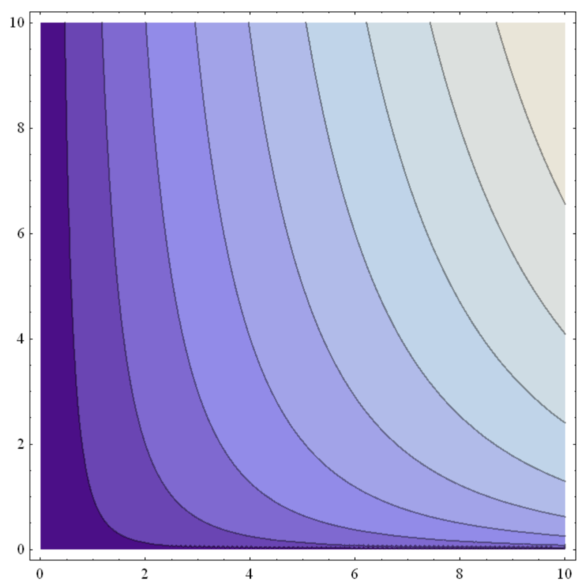


Figure 7. On the left: contour graph of a function satisfying Chandler's form specifically $U(x, y) = \frac{2}{3} * \sqrt{x} + \frac{1}{3} * \sqrt{y}$. On the right: contour graph of a function in the Cobb-Douglas form, specifically $U(x, y) = x^{3/4}y^{1/4}$.

Like the Cobb-Douglas form, this sum of square roots produces a functional form which is monotonically increasing in all directions and allows for a set of convex isoquants. Interestingly, the results of changing the coefficients of these square roots has, at first glance, a similar effect to manipulating the powers of x and y in the Cobb-Douglas model. As seen in Figure 7, one of the differences between these two approaches is that the Cobb-Douglas functional form seems likely to yield lower utility for highly imbalanced bundles than the form used by Chandler. The placement of contours in the Cobb-Douglas form are such that they tend to touch neither axis but rather asymptotically approach them, whereas in Chandler's form, contours appear to actually touch at least one axis for some finite quantity owned of some good.

Simulating Market Scenarios with an Incorrectly Estimated Preference Structure

As a result of difficulties arising from quantitative interpretation of Chandler's visual style of output, and its lack of an option for numeric inputs, it becomes necessary to simply reason mathematically to anticipate the differences between a perfectly free-market scenario where all parties are fully informed and one in which asymmetric information is a non-negligible problem. One way of modeling the emergence of imperfect information would be to assume at least one party incorrectly estimates their marginal utilities of exchange; then the expected result might deviate from the ideal one, so that the partially uninformed individual might gain less from trade as the result of either the second party's willingness to exploit the first party's ignorance, or as a result of the second party's ignorance of the first party's ignorance.

To examine this phenomenon, assume both individuals' true consumer preferences are governed by two-dimensional Cobb-Douglas functions of the form $U_1(x, y) = x^{\alpha_1} y^{1-\alpha_1}$ and $U_2(x, y) = x^{\alpha_2} y^{1-\alpha_2}$, which as oulined earlier satisfy the convexity properties required to satisfy the Second Welfare Theorem. Then to set up the Edgeworth box, the second individual's utility function must be transformed so that its contours are translated by the vector of total quantities of each good produced (Q_x, Q_y), due to the functional form's symmetry with respect to the origin. Then the resulting functional form is $U_2(x, y) = -(x - Q_x)^{\alpha_2} (y - Q_y)^{1-\alpha_2}$ (with the negative sign included to ensure that the utility is positive and increasing as *x* and *y* decrease, since *x* and *y* correspond to the amount $\boldsymbol{A_1}$ owned by the first individual). Then by the tangency definition of the contract curve, the contract curve must then be the real solution to the differential equation

$\frac{\frac{\partial}{\partial \mathrm{x}}(x^{\alpha_2}y^{1-\alpha_2})}{\frac{\partial}{\partial \mathrm{y}}(x^{\alpha_2}y^{1-\alpha_2})} = \frac{\frac{\partial}{\partial \mathrm{x}}(-(x-Q_x)^{\alpha_2}(y-Q_y)^{1-\alpha_2})}{\frac{\partial}{\partial \mathrm{y}}(-(x-Q_x)^{\alpha_2}(y-Q_y)^{1-\alpha_2})}$, which is, according to Mathematica,

$y = \frac{Q_y \cdot x(-1+\alpha_1)\cdot\alpha_2}{-Q_x\cdot\alpha_1 + x\alpha_1 - x\alpha_2 + Q_x\alpha_1\alpha_2}$. Assuming an exhaustive distribution of goods (i.e., no surplus), where the first party owns (*p*,*q*) and the second owns $(Q_x - p, Q_y - q)$, the set of available Pareto-optimal allocations will be bounded by the intersection of the contract curve

$\{(x,y): y = \frac{Q_y \cdot x(-1+\alpha_1)\cdot\alpha_2}{-Q_x\cdot\alpha_1 + x\alpha_1 - x\alpha_2 + Q_x\alpha_1\alpha_2}\}$ and the convex subregion of the Edgeworth box bounded by isoquants $\{(x,y): U_1(x,y) = p^{\alpha_1}q^{1-\alpha_1}\}$ and $\{(x,y): U_2(x,y) = -(p-Q_x)^{\alpha_2}(q-Q_y)^{1-\alpha_2}\}$, which are the isoquants intersecting at (*p*,*q*). Then each bound of the relevant segment of the contract curve can be found by solving $\sqrt[(1-\alpha_1)]{\frac{p^{\alpha_1}q^{1-\alpha_1}}{x^{\alpha_1}}} = \frac{Q_y \cdot x(-1+\alpha_1)\cdot\alpha_2}{-Q_x\cdot\alpha_1 + x\alpha_1 - x\alpha_2 + Q_x\alpha_1\alpha_2}$ and

$\sqrt[(1-\alpha_2)]{\frac{p^{\alpha_2}q^{1-\alpha_2}}{x^{\alpha_2}}} = \frac{Q_y \cdot x(-1+\alpha_1)\cdot\alpha_2}{-Q_x\cdot\alpha_1 + x\alpha_1 - x\alpha_2 + Q_x\alpha_1\alpha_2}$, respectively (unfortunately, Mathematica is not able to solve either of these general equations since the form of the solution depends on α_1). In Figure 8, the two points intersecting the contract curve (dark yellow) and each of the isoquants (blue and purple) are the real solutions to two special cases of the aforementioned two equations. If the final reallocation ends up on point A, then the second individual's gains are maximized within the set of Pareto improvements (light blue) and the first individual gains nothing, and if the final reallocation ends up on point B, the first individual's gains are maximized and the second individual gains nothing. Regardless of how much each party gains, a Pareto-optimal allocation must lie somewhere on the segment of the contract curve between points A and B (orange).

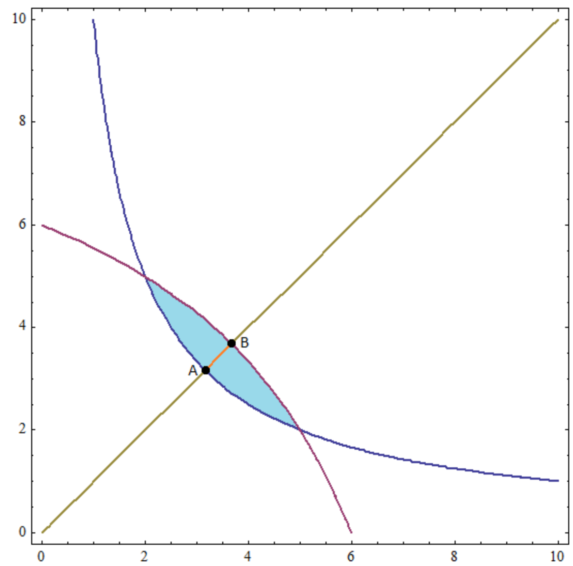


Figure 8. Two isoquants, forming the boundaries of the set of Pareto improvements and part of the contract curve.

Now, presume the first party is for some reason sufficiently ignorant of their own utility that the isoquant $\{(x,y)\colon U_1(x,y) = p^{\alpha_1}q^{1-\alpha_1}\}$ is incorrectly estimated to be an isoquant determined by some transformation of the correct Cobb-Douglas form. A problem quickly arises when one simply tries to change the parameter α_1 while keeping p, q, and thus the current level of utility $U_1 = p^{\alpha_1}q^{1-\alpha_1}$ fixed; as it turns out, for any α_1, a_1, the only solution to the system of equations $x^{\alpha_1}y^{1-\alpha_1} = U_1$, $x^{a_1}y^{1-a_1} = U_1$ is $\alpha_1 = a_1 = 0$. Then the transformation warrants an additional step, involving addition of the inputs *x* and *y* by arbitrarily chosen quantities *m* and *n* (though with the restriction that $(p+m, q+n) \in [0, Q_x] \times [0, Q_y]$; this is necessary to keep the ordered pair inside the relevant domain). So the false first utility function becomes $V_1(x,y) = (x+m)^{a_1}(y+n)^{1-a_1}$, such that $V_1(p,q) = U_1(p,q)$ and $0 \le a_1 \le 1$, reflecting an assumption that in order for the first party to be unaware that they are incorrect about their true preference, they must yield the same level of utility from possessing (p,q) as designated by both the correct and incorrect function. An immediate consequence of setting $V_1(p,q) = U_1(p,q)$ is that a specific value can usually be found for a_1; solving the aforementioned equality for a_1 yields $a_1 = \frac{\ln(p^{\alpha_1}q^{1-\alpha_1}) - \ln(n+q)}{\ln(m+p) - \ln(n+q)}$, implying that if $p + m = q + n$ or either $p^{\alpha_1}q^{1-\alpha_1} = 0$, $p + m = 0$, or $q + n = 0$, then this transformation will not work, as a_1 will be undefined.

Depending on which values of *m* and *n* are chosen, the first party will either underestimate or overestimate the boundary of the convex trading space closer to that individual's "origin" (the corner of the Edgeworth box for which that individual has nothing). In a situation where the second party is aware of the first party's ignorance, depending on the nature of the first party's ignorance this can provide either an advantage or a disadvantage to the second party during negotiation. If the first party believes their isoquant to have a curve further from the origin than it really is, but the second party knows where it really lies, then there are a few different possibilities: if the second party is honest and establishes the true location of their isoquant, either the available trading space shrinks so that the ignorant party can grant itself leverage by insisting on maintaining their established lower bound, or the trading space is restricted to the starting point, where each party is made to believe that mutually beneficial trade is impossible since they cannot agree on the correct location of the contract curve. While the true contract curve is defined as the values of (*x*,*y*) for which $y = \frac{Q_y \cdot x(-1+\alpha_1)\cdot\alpha_2}{-Q_x \cdot \alpha_1 + x\alpha_1 - x\alpha_2 + Q_x\alpha_1\alpha_2}$, the incorrect one that the first party would have to believe is correct (to remain consistent) is found by solving

$$\frac{\frac{\partial}{\partial \mathrm{x}}\left((x+m)^{\frac{\ln(p^{\alpha_1}q^{1-\alpha_1})-\ln(n+q)}{\ln(m+p)-\ln(n+q)}}(y+n)^{1-\frac{\ln(p^{\alpha_1}q^{1-\alpha_1})-\ln(n+q)}{\ln(m+p)-\ln(n+q)}}\right)}{\frac{\partial}{\partial \mathrm{y}}\left((x+m)^{\frac{\ln(p^{\alpha_1}q^{1-\alpha_1})-\ln(n+q)}{\ln(m+p)-\ln(n+q)}}(y+n)^{1-\frac{\ln(p^{\alpha_1}q^{1-\alpha_1})-\ln(n+q)}{\ln(m+p)-\ln(n+q)}}\right)} = \frac{\frac{\partial}{\partial \mathrm{x}}(-(x-Q_x)^{\alpha_2}(y-Q_y)^{1-\alpha_2})}{\frac{\partial}{\partial \mathrm{y}}(-(x-Q_x)^{\alpha_2}(y-Q_y)^{1-\alpha_2})},$$ whose solution

becomes $$y = \frac{-\frac{Q_y\alpha_2}{(-Q_x+x)(1-\alpha_2)} - \frac{n(ln[p^{\alpha_1}q^{1-\alpha_1}]-ln[n+q])}{(m+x)\left(1-\frac{ln[p^{\alpha_1}q^{1-\alpha_1}]-ln[n+q]}{ln[m+p]-ln[n+q]}\right)(ln[m+p]-ln[n+q])}}{-\frac{\alpha_2}{(-Q_x+x)(1-\alpha_2)} + \frac{ln[p^{\alpha_1}q^{1-\alpha_1}]-ln[n+q]}{(m+x)\left(1-\frac{ln[p^{\alpha_1}q^{1-\alpha_1}]-ln[n+q]}{ln[m+p]-ln[n+q]}\right)(ln[m+p]-ln[n+q])}}.$$

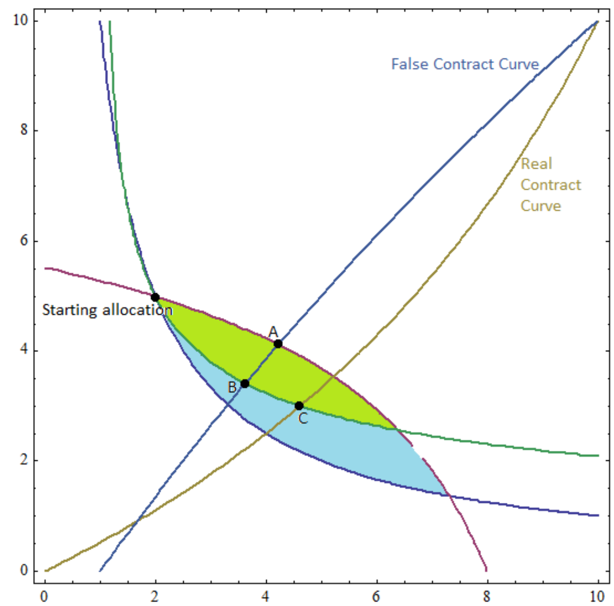


Figure 9. One party incorrectly underestimates their potential gains.

Figure 9 features an example of a scenario where one individual, whose "origin" is on the bottom left believes their isoquant to be further from the bottom left corner than it is (the true isoquant here is blue, the false one green). As a result, this first individual will believe the set of Pareto improvements (lime green) to be a subset of what it really is (lime green and sky blue). If the second individual is unaware that the first is incorrectly communicating their preference structure, then Pareto-efficient negotiation must occur within the smaller trading space, and the final allocation must lie somewhere on the false contract curve, between points A and B; this guarantees a net gain for the first individual that they would not know about. If the second individual knows the first individual's real preference structure, then they can choose to either inform the first person of their true preference structure (thereby establishing the true contract curve), or negotiate under the pretense of symmetric information. In this case, being honest might backfire for the second individual, because the first might insist on setting their minimum Pareto-optimal allocation at point C as a result of some irrational cognitive bias (similar to the "anchoring" bias described by Kahneman and Taversky). For the initially misinformed individual, ending up in the blue region feels like a "loss" due to having the green isoquant as an initial reference point. Point B, meanwhile, appears to yield a higher maximum gain for the second party, even if it isn't truly Pareto-optimal; then it might be in the interest of the second party to acknowledge as correct the delusions of the first.

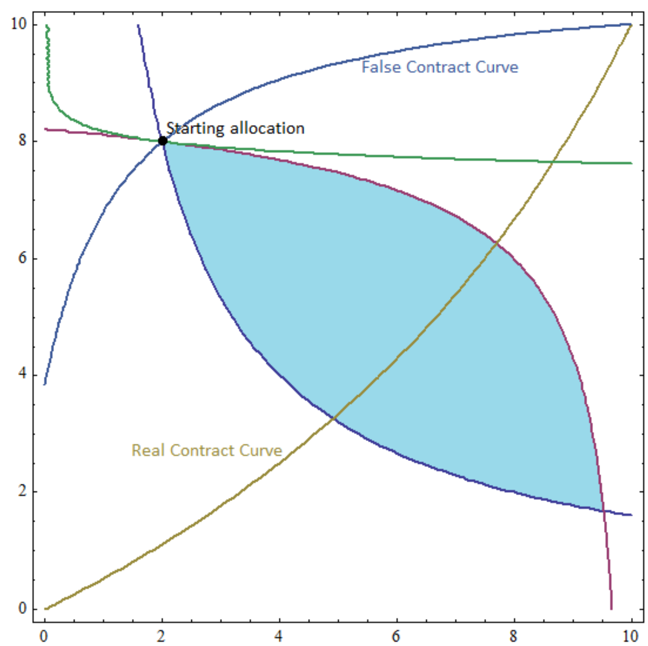


Figure 10. The first party mistakenly believes the initial allocation to be Pareto-optimal.

In an extreme case, as seen in Figure 10, the first party might mistake an initial allocation for a Pareto-optimal one. In the example below, the first party greatly has most of *y* but little of *x*, and greatly underestimates the value of additional units of *x* in terms of units of *y*, mistakenly assuming that this is an acceptable market price level. The second party, meanwhile, who has most of *x* but little of *y*, recognizes that both parties stand to gain a lot through trading (with the region of Pareto improvements being considerably large). In this case the second party's only rational option would be to attempt to convince the first party that they have incorrectly estimated their own preference structure. If the first party disagrees and is stubborn, the second party has no hope of improving their utility.

Alternatively, the ignorance of the first party can be a boon for the second; in a scenario where the ignorant party underestimates the quantity lying on the contract curve to which they would be indifferent to with respect to the initial allocation, the second party, if unaware of their leverage, could easily negotiate with the first party so that the first actually ends up worse off than initially (the same result would be expected if the second party knew the first party's true indifference curve, and then out of self-interest exploited this fact, especially if the false contract curve generated by the first party's ignorance yields a much higher potential net gain for the second party). Of course, in this scenario Pareto-efficient trade, and in turn Pareto optimization, are still possible and plausible; nevertheless it is not guaranteed. For instance, Figure 11 presents a scenario in which the first individual greatly overestimates potential gains; the set of true Pareto improvements is limited to the light blue space, but the first party mistakes the pink region as being part of the set of Pareto improvements. As mentioned already, if the second party is behaving in their own self-interest first and foremost, they will trade with the first party under the pretense of Pareto optimization, using the false contract curve. The result is that the set of final reallocations will lie on the region of the contract curve between points B and C, with almost this entire segment lying in the pink region. Only the reallocations between points C and

D entail true gains for both parties, but the true gains are marginal for both parties; therefore, since the second party has the opportunity to gain much more, it is unlikely for the final reallocation to end up there. If the second party knows the first party to be incorrect, the opportunity cost of being truthful (in other words, the cost of not lying) is substantial enough for it to be almost foolish to forgo potentially reaching point B rather than limiting maximum gains to point A.

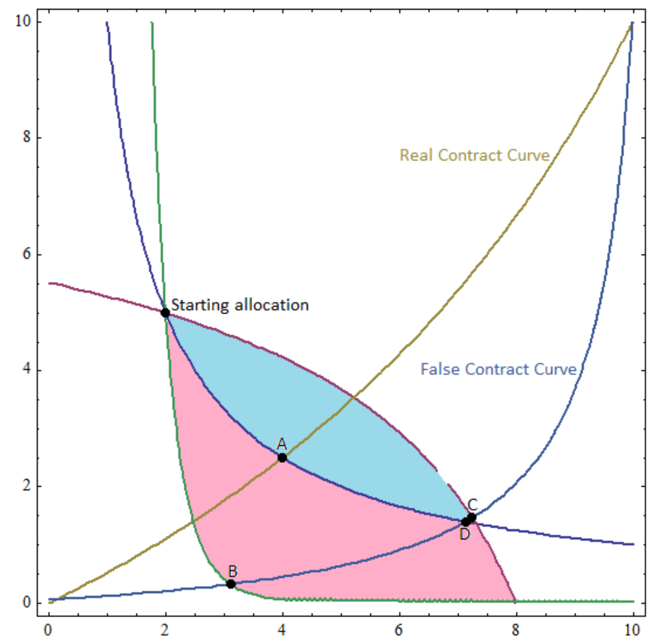


Figure 11. The first party suffers the risk of bearing losses under the false pretense that they are real gains.

Conclusions; Closing Remarks

Since relative price levels between goods are determined by the tangential intersections of indifference curves corresponding to each point on the contract curve, incorporating inaccurate information about one party's own preference structure will change the way prices are set. Thus the range of possible prices that can be set is changed as well. This is most obvious in the case where at least one party believes the starting allocation to already be Pareto-optimal; this leads that individual to draw the false conclusion that the resulting set of relative prices are fair market equilibrium prices, when demand from other individuals might suggest otherwise. The resulting market failure demonstrates that the First Welfare Theorem is not merely vacuously true; while the starting allocation behaves effectively as a market equilibrium, it is not truly Pareto-optimal.

Based on the aforementioned definition of nominal GDP (that is, the sum of products of unit price and quantity produced), differences in the final price levels from truly Pareto-optimal ones will result in very different measures of nominal GDP. In a simulation incorporating consumption and production cycles, the resulting difference in allocation will change how each agent produces additional units in the future; in the bi-variable case, if for instance one party had more of x and less of y, then assuming that party did not specialize in production of x, they would be more likely to produce y due to its relative scarcity. Meanwhile, if trade is restricted due to one's insistence on the existence of fewer Pareto-efficient reallocations than there are, nominal GDP (at least at that time) could be greatly distorted, possibly to the point of broadcasting expansion when recession is truly occurring. If periodic inflation could be tracked, one could more accurately observe just how much the economy either prospers or suffers relative to a situation where information is not problematic and trade is thus always truly Pareto-efficient. On the other hand, cases where one individual becomes wealthy and the other poor as a result of bad information are unlikely to be seen as problematic from the aggregative perspective (since all quantities owned are added up regardless of distribution). However, wealth disparity can very quickly emerge in the situations where one party is made worse off due to incorrectly identifying detrimental reallocations as mutual marginal gains.

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